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Amplitude modulation of electromagnetic waves by alternating magnetic fields

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Abstract. In this paper the authors have given a quantitative analytical investigation of the interesting concept of the modulation of an electromagnetic wave by its propagation along an alternating magnetic field in a semiconductor or a plasma. Numerical results, presented at the end, show that this phenomenon is appreciable.

1. Introduction

Mason *et al.* (1953) have shown that, when a crystal is simultaneously subjected to a mutually perpendicular alternating electric field of frequency ω and alternating magnetic field of frequency Ω, the Hall voltage alternates with the frequency ω and the sideband frequencies $\omega \pm \Omega$. This suggests that an electromagnetic wave, propagating in a semiconductor or a plasma along the direction of an alternating magnetic field, should become amplitude modulated at the frequency of the magnetic field. This paper presents a quantitative investigation of this interesting concept.

The Boltzmann transfer equation for free electrons has been solved by the authors in the presence of mutually perpendicular alternating electric and magnetic fields. Various time-dependent components of the distribution function have been evaluated by solving the coupled system of equations and an expression for the time-dependent current density has been derived; the only assumption used is that the magnitudes of the electric vector of the generated frequencies are much smaller than that of the incident frequency. The expression for current density has been substituted in the general wave equation, and the solutions have been used for investigating the magnitude of modulation of the wave after it propagates a certain distance in the medium and also of the reflected wave from the medium–free-space interface.

Some numerical calculations have been made for semiconductors as well as plasmas when the incident frequency is equal to the gyrofrequency of the electrons ($\omega_B = eB/mc$); the results have been presented in the form of tables.

2. Sideband components in the current density

The Boltzmann transfer equation for electrons in a homogeneous medium subjected to an electric field $\mathbf{E} = \mathbf{E}_1 \exp(i\omega t)$ in the xy plane and a magnetic field $B = B_0 + B_1 \exp(i\Omega t)$ in the z direction is given by

$$\frac{\partial f}{dt} + \alpha_x' \frac{\partial f}{\partial v_x} + \alpha_y' \frac{\partial f}{\partial v_y} = \left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}} \tag{2.1}$$

where

$$\alpha_x' = -[a_{x_1} \exp(i\omega t) + v_y\{\omega_B + \omega_{B_1} \exp(i\Omega t)\}] \tag{2.2}$$

$$\alpha_y' = -[a_{y_1} \exp(i\omega t) - v_x\{\omega_B + \omega_{B_1} \exp(i\Omega t)\}] \tag{2.3}$$

$$\alpha_{x_1} = eE_{1x}/m, \qquad a_{y_1} = eE_{1y}/m$$

ω is the frequency of the wave, Ω that of the applied magnetic field, $\omega_B = eB_0/mc$ and $\omega_{B_1} = eB_1/mc$; $(\partial f/\partial t)_{\mathrm{c}}$ represents the rate of change of the distribution function due to collisions which for Lorentzian plasmas and simple-model non-degenerate semiconductors with dominant acoustic scattering is given by the expression (Desloge and Matthysse 1960)

$$\left(\frac{\partial f}{\partial t}\right)_{c} = -\nu(f-f_0)+\frac{m}{Mv^2}\frac{\partial}{\partial v}(\nu v^3 f_0)+\frac{kT}{Mv^2}\frac{\partial}{\partial v}\left(\nu v^2\frac{\partial f_0}{\partial v}\right) \tag{2.4}$$

where the symbols used have their usual meaning. In the present context the distribution function of electron velocities may be expanded in the form

$$\begin{aligned} f = f_0 &+ v_x[f_{x_1}{}^1\exp(i\omega t)+f_{x_1}{}^2\exp\{i(\omega+\Omega)t\}+f_{x_1}{}^3\exp\{i(\omega-\Omega)t\}] \\ &+ v_y[f_{y_1}{}^1\exp(i\omega t)+f_{y_1}{}^2\exp\{i(\omega+\Omega)t\}+f_{y_1}{}^3\exp\{i(\omega-\Omega)t\}]. \end{aligned} \tag{2.5}$$

Substituting for α_x', α_y', $(\partial f/\partial t)_c$ and f from equations (2.2) to (2.5) in equation (2.1), and equating the coefficients of $v_{x,y}\exp(i\omega t)$, $v_{x,y}\exp\{i(\omega+\Omega)t\}$ and $v_{x,y}\exp\{i(\omega-\Omega)t\}$ on both sides of the resulting equation, one obtains a set of coupled equations involving the various components of f. It may be mentioned here that while performing the product of terms involving complex exponentials, one must keep in mind that the real part of the product is not equal to the product of the real parts of the two multiplying quantities. Thus one has to modify the simple product operation to the form

$$\begin{aligned} &\mathscr{R}\{A_1\exp(i\omega_1 t)\}\mathscr{R}\{A_2\exp(i\omega_2 t)\} \\ &\quad = \tfrac{1}{2}\mathscr{R}[A_1A_2\exp\{i(\omega_1+\omega_2)t\}+A_1\tilde{A}_2\exp\{i(\omega_1-\omega_2)t\}] \end{aligned}$$

where $\mathscr{R}$ denotes the real part and $\tilde{A}_2$ denotes the complex conjugate of A_2; A_1, A_2 have been chosen complex to allow for any phase terms in the multiplying quantities.

By solving the above-mentioned set of coupled equations, using the approximation that $f_{x_1}{}^2, f_{x_1}{}^3 \ll f_{x_1}{}^1$ and $f_{y_1}{}^2, f_{y_1}{}^3 \ll f_{y_1}{}^1$ (which means that the generated sideband components have a much smaller magnitude than the fundamental component), expressions for various components of the distribution function may be derived; these are

$$\begin{aligned} f_{x_1}{}^1 = {} & [\nu^3 a_{x_1}-\nu^2(\omega_B a_{y_1}+i\omega a_{x_1}+\nu\{(\omega_B{}^2+\omega^2)a_{x_1}+2i\omega\omega_B a_{y_1}\}+(\omega_B{}^2-\omega^2)(i\omega a_{x_1}-\omega_B a_{y_1})] \\ & \times\frac{1}{v}\frac{\partial f_0}{\partial v}[\{\nu^2+(\omega+\omega_B)^2\}\{\nu^2+(\omega-\omega_B)^2\}]^{-1} \end{aligned} \tag{2.6}$$

$$\begin{aligned} f_{x_1}{}^2 = {} & -\frac{\omega_{B_1}}{2}(\nu^6 a_{y_1}+A_{x_5}\nu^5\omega_0+A_{x_4}\nu^4\omega_0{}^2+A_{x_3}\nu^3\omega_0{}^3+A_{x_2}\nu^2\omega_0{}^4+A_{x_1}\nu\omega_0{}^5+A_{x_0}\omega_0{}^6)\frac{1}{v}\frac{\partial f_0}{\partial v} \\ & \times[\{\nu^2+(\omega+\omega_B)^2\}\{\nu^2+(\omega-\omega_B)^2\}\{\nu^2+(\omega+\Omega+\omega_B)^2\}\{\nu^2+(\omega+\Omega-\omega_B)^2\}]^{-1} \end{aligned} \tag{2.7}$$

where ω_0 is an arbitrary normalizing frequency and

$$A_{x_5} = \frac{1}{\omega_0}\{2\omega_B a_{x_1}-i(2\omega+\Omega)a_{y_1}\} \tag{2.8a}$$

$$A_{x_4} = \frac{1}{\omega_0{}^2}[\{\omega_B{}^2-\omega^2-(\omega+\Omega)(6\omega+\Omega)+2(2\omega+\Omega)^2\}a_{y_1}-3i\omega_B(2\omega+\Omega)a_{x_1}] \tag{2.8b}$$

$$A_{x_3} = \frac{1}{\omega_0{}^3}[4\omega_B\{\omega_B{}^2-\omega(\omega+\Omega)\}a_{x_1}+i(2\omega+\Omega)\{2\omega_B{}^2-\omega^2-(\omega+\Omega)^2\}a_{y_1}] \tag{2.8c}$$

$$\begin{aligned} A_{x_2} = {} & \frac{1}{\omega_0{}^4}[\{\omega_B{}^2-(\omega+\Omega)^2\}(\omega_B{}^2-\omega^2)a_{y_1} \\ & -\{\omega_B{}^2+\omega(\omega+\Omega)\}\{2\omega_B{}^2-\omega^2-(\omega+\Omega)(5\omega+\Omega)\}a_{y_1} \\ & +2(2\omega+\Omega)^2\{\omega_B{}^2-\omega(\omega+\Omega)\}a_{y_1}-i\{2\omega_B{}^2+\omega^2+(\omega+\Omega)^2\}(2\omega+\Omega)\omega_B a_{x_1}] \end{aligned} \tag{2.8d}$$

$$A_{x_1} = \frac{1}{\omega_0^5}[2\omega_B(\omega_B^2-\omega^2)\{\omega_B^2-(\omega+\Omega)^2\}a_{x_1}+2\omega_B(2\omega+\Omega)^2\{\omega_B^2-\omega(\omega+\Omega)\}a_{x_1}$$
$$+i(2\omega+\Omega)(\omega_B^2-\omega^2)\{\omega_B^2-(\omega+\Omega)^2\}a_{y_1}+2i(2\omega+\Omega)\{\omega_B^4-\omega^2(\omega+\Omega)^2\}a_{y_1} \tag{2.8e}$$

$$A_{x_0} = \frac{1}{\omega_0^6}[-(\omega_B^2-\omega^2)\{\omega_B^2-(\omega+\Omega)^2\}\{\omega_B^2-\omega(\omega+\Omega)\}a_{y_1}$$
$$+i\omega_B(2\omega+\Omega)\{\omega_B^2-(\omega+\Omega)^2\}(\omega_B^2-\omega^2)a_{x_1}]. \tag{2.8f}$$

Expressions for the y components of the distribution function can be written similarly by replacing a_{x_1} by a_{y_1} and a_{y_1} by $-a_{x_1}$. The expressions corresponding to the frequency $\omega-\Omega$ can be written by replacing Ω by $-\Omega$ in equation (2.7).

In the following analysis the form of the isotropic part of the electron distribution function has been taken to be Maxwellian; this means that the non-linear contribution due to external fields has been assumed to be negligibly small. Thus we assume

$$f_0 = N\left(\frac{m}{2\pi kT}\right)^{3/2}\exp\left(-\frac{mv^2}{2kT}\right) \tag{2.9}$$

where the various symbols have their usual meanings.

The expression for the current-density components can be derived by substituting the values of components of f in the equation

$$J = -e\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{v}f\,dv_x\,dv_y\,dv_z$$

which, using equation (2.5), can be simplified to the form

$$J = -\frac{4\pi e}{3}\int_0^\infty v^4 f_1\,dv. \tag{2.10}$$

The expression for the current-density components can be written in a convenient form (so as to correspond to the two modes of propagation of an electromagnetic wave)

$$J_{x_1}{}^1 \pm iJ_{y_1}{}^1 = A_\pm(E_{x_1}\pm E_{y_1})\exp(i\omega t) \tag{2.11a}$$
$$J_{x_1}{}^2 \pm iJ_{y_1}{}^2 = C_\pm(E_{x_1}\pm iE_{y_1})\exp\{i(\omega+\Omega)t\} \tag{2.11b}$$

where $A_\pm$ and $C_\pm$ are complex coefficients involving integrals. These constants have been evaluated in two different cases taking a general dependence of the collision frequency on energy

$$\nu = \nu_0 x^{n/2}$$

where $x = mv^2/2kT$ is the dimensionless kinetic energy of electrons.

2.1. *Case I*

The collision frequency is much larger than the wave frequency, i.e. $\nu \gg \omega$. Then

$$A_\pm = 2B\left\{\frac{1}{\nu_0}\prod\left(\frac{3-n}{2}\right)-i\frac{(\omega\mp\omega_B)}{\nu_0^2}\prod\left(\frac{3-2n}{2}\right)\right\} \tag{2.12a}$$

$$C_\pm = -B\frac{\omega_{B_1}}{\omega^2}\left[\frac{2\omega^2\omega_{B_1}}{\nu_0^3}\prod\left(\frac{3-3n}{2}\right)\mp\left\{i\frac{\omega^2}{\nu_0^2}\prod\left(\frac{3-2n}{2}\right)+\frac{(2\omega+\Omega)\omega^2}{\nu_0^3}\prod\left(\frac{3-3n}{2}\right)\right\}\right]. \tag{2.12b}$$

2.2. *Case II*:

In this case the collision frequency is much smaller than the wave frequency, i.e. $\nu \ll \omega$. We further distinguish between the following three cases:

(i) When $\omega = \omega_B$ and $\nu > \Omega$ (at gyrofrequency of the medium)

$$A_+ = \frac{2B}{\nu_0}\prod\left(\frac{3-n}{2}\right) \tag{2.13a}$$

$$A_- = -i\frac{B}{\omega}\prod\left(\frac{3}{2}\right) \tag{2.13b}$$

$$C_+ = 2B\frac{\omega_{B_1}}{\omega^2}\left\{i\frac{\omega^2}{\nu_0^2}\prod\left(\frac{3-2n}{2}\right) + \frac{\Omega}{\omega}\frac{\omega^3}{\nu_0^3}\prod\left(\frac{3-3n}{2}\right)\right\} \tag{2.13c}$$

$$C_- = 0 \tag{2.13d}$$

(ii) At low magnetic fields, i.e. $\omega \gg \omega_B$,

$$A_\pm = \frac{2B}{\omega}\left\{-i\prod\left(\frac{3}{2}\right) + \frac{\nu_0}{\omega}\prod\left(\frac{3+n}{2}\right) \mp i\frac{\omega_B}{\omega}\prod\left(\frac{3}{2}\right)\right\} \tag{2.14a}$$

$$C_\pm = B\frac{\omega_{B_1}}{\omega^2}\left[2\frac{\omega_B}{\omega}\left\{3\frac{\nu_0}{\omega}\prod\left(\frac{3+n}{2}\right) - i\prod\left(\frac{3}{2}\right)\right\} \pm \left\{2\frac{\nu_0}{\omega}\prod\left(\frac{3+n}{2}\right) - i\prod\left(\frac{3}{2}\right)\right\}\right]. \tag{2.14b}$$

(iii) At high magnetic fields, i.e. $\omega \ll \omega_B$,

$$A_\pm = \frac{2B}{\omega_B}\left[\left\{\frac{\nu_0}{\omega_B}\prod\left(\frac{3+n}{2}\right) + i\frac{\omega}{\omega_B}\prod\left(\frac{3}{2}\right)\right\} \pm i\prod\left(\frac{3}{2}\right)\right] \tag{2.15a}$$

$$C_\pm = -B\frac{\omega_{B_1}}{\omega_B^2}\left[2\left\{\frac{\nu_0}{\omega_B}\prod\left(\frac{3+n}{2}\right) + i\frac{\omega}{\omega_B}\prod\left(\frac{3}{2}\right)\right\} \pm \left\{6\frac{\omega}{\omega_B}\frac{\nu_0}{\omega_B}\prod\left(\frac{3+n}{2}\right) + i\prod\left(\frac{3}{2}\right)\right\}\right] \tag{2.15b}$$

where

$$B = \frac{e^2N}{3m\prod(\frac{1}{2})}$$

and

$$\prod(r) = \int_0^\infty x^r\, e^{-x}\, dx = \Gamma(r+1).$$

The expression for the total current density including the component due to the generated fields $E_{x_2} \pm iE_{y_2}$ is given by

$$J_{x_1}{}^2 \pm iJ_{y_1}{}^2 = A_\pm{}'(E_{x_2} \pm iE_{y_2}) + C_\pm(E_{x_1} \pm iE_{y_1}) \tag{2.16}$$

where $A_\pm{}'$ have a similar expression to those of $A_\pm$ except for the difference that ω is replaced by $\omega+\Omega$. The rest of the expressions for $J_{x_1}{}^3 \pm iJ_{y_1}{}^3$ can be written by symmetry by replacing Ω by $-\Omega$ in expressions for $J_{x_1}{}^2 \pm iJ_{y_1}{}^2$.

3. Modulation in propagating and reflected waves

The wave equations governing the propagation of the two modes of electromagnetic wave along the z direction in a medium with free carriers (in the presence of a magnetic field) may be written as

$$\frac{d^2(E_x \pm iE_y)}{dz^2} = \frac{\epsilon\mu}{c^2}\frac{\partial^2(E_x \pm iE_y)}{\partial t^2} + \frac{4\pi\mu}{c^2}\frac{\partial(J_x \pm iJ_y)}{\partial t} \tag{3.1}$$

which may be reduced to the dimensionless form

$$\frac{\partial^2(\mathscr{E}_x \pm i\mathscr{E}_y)}{\partial \xi^2} = \frac{1}{\omega^2}\frac{\partial^2(\mathscr{E}_x \pm i\mathscr{E}_y)}{\partial t^2} + \frac{4\pi}{\epsilon\omega^2}\frac{\partial\{(J_x \pm iJ_y)/E_{00}\}}{\partial t} \tag{3.2}$$

where $\mathscr{E}_x = E_x/E_{00}$, E_{00} being any arbitrary normalizing field, $\xi = (\epsilon\mu)^{1/2}2\pi z/\lambda$, ϵ is the d.c. dielectric constant of the medium and μ the magnetic permeability. Substituting the expressions for the current density components from equations (2.11*a*) and (2.16) in equation (3.2), the wave equations corresponding to different frequencies come out to be

$$\frac{\partial^2(\mathscr{E}_{x_1} \pm i\mathscr{E}_{y_1})}{\partial \xi^2} + (\beta_1{}^{\pm})^2(\mathscr{E}_{x_1} \pm i\mathscr{E}_{y_1}) = 0 \tag{3.3}$$

$$\frac{\partial^2(\mathscr{E}_{x_2} \pm i\mathscr{E}_{y_2})}{\partial \xi^2} + (\beta_2{}^{\pm})^2(\mathscr{E}_{x_2} \pm i\mathscr{E}_{y_2}) = \delta_2{}^{\pm}(\mathscr{E}_{x_1} \pm i\mathscr{E}_{y_1}) \tag{3.4}$$

$$\frac{\partial^2(\mathscr{E}_{x_3} \pm i\mathscr{E}_{y_3})}{\partial \xi^2} + (\beta_3{}^{\pm})^2(\mathscr{E}_{x_3} \pm i\mathscr{E}_{y_3}) = \delta_3{}^{\pm}(\mathscr{E}_{x_1} \pm i\mathscr{E}_{y_1}) \tag{3.5}$$

where the various constants are given by

$$(\beta_1{}^{\pm})^2 = 1 - \frac{4\pi i}{\epsilon\omega^2}\omega A_{\pm} \tag{3.6a}$$

$$(\beta_2{}^{\pm})^2 = \left(\frac{\omega+\Omega}{\omega}\right)^2 - \frac{4\pi i}{\epsilon\omega^2}(\omega+\Omega)A'_{\pm} \tag{3.6b}$$

$$\delta_2{}^{\pm} = \frac{4\pi i}{\epsilon\omega^2}(\omega+\Omega)C_{\pm} \tag{3.6c}$$

$(\beta_3{}^{\pm})^2$ and $\delta_3{}^{\pm}$ can be written by replacing Ω by $-\Omega$ in the expressions for $(\beta_2{}^{\pm})^2$ and $\delta_2{}^{\pm}$ respectively.

The solution of the equation (3.3) is

$$\mathscr{E}_{x_1} \pm i\mathscr{E}_{y_1} = K_1{}^{\pm}\exp(i\beta_1{}^{\pm}\xi) \tag{3.7}$$

where the second term vanishes owing to the radiation condition, viz. that the field vectors vanish at $\xi = \infty$.

Substituting this value of electric field vectors from equations (3.7) in equations (3.4) we obtain

$$\frac{\partial^2(\mathscr{E}_{x_2} \pm i\mathscr{E}_{y_2})}{\partial \xi^2} + (\beta_2{}^{\pm})^2(\mathscr{E}_{x_2} \pm i\mathscr{E}_{y_2}) = \delta_2{}^{\pm}K_1{}^{\pm}\exp(i\beta_1{}^{\pm}\xi). \tag{3.8}$$

This can be solved explicitly for two different cases:

(i) When $\beta_2{}^{\pm} \neq \beta_1{}^{\pm}$, the solution is

$$\mathscr{E}_{x_2} \pm i\mathscr{E}_{y_2} = K_2{}^{\pm}\exp(i\beta_2{}^{\pm}\xi) + \frac{\delta_2{}^{\pm}K_1{}^{\pm}\exp(i\beta_1{}^{\pm}\xi)}{(\beta_2{}^{\pm})^2 - (\beta_1{}^{\pm})^2} \tag{3.9}$$

where $K_2{}^{\pm}$ is the constant of integration and is to be evaluated separately for the cases of propagation and reflection.

(ii) When $\beta_2{}^{\pm} = \beta_1{}^{\pm}$ (this case corresponds to $\omega \gg \Omega$), the solution is of the form

$$\mathscr{E}_{x_2} \pm i\mathscr{E}_{y_2} = \left(\frac{\delta_2{}^{\pm}K_1{}^{\pm}}{2i\beta_1{}^{\pm}}\xi + \frac{\delta_2{}^{\pm}K_1{}^{\pm}}{4(\beta_1{}^{\pm})^2} + K_2{}^{\pm}\right)\exp(i\beta_1{}^{\pm}\xi). \tag{3.10}$$

Here again $K_2{}^{\pm}$ are arbitrary constants as in (i).

3.1. *Propagation*

If we apply the boundary condition that

$$\mathscr{E}_{x_2} \pm i\mathscr{E}_{y_2} = 0 \quad \text{at} \quad \xi = 0$$

the amplitude of the generated wave of frequency $\omega + \Omega$ at a distance ξ is

$$\mathscr{E}_{x_2} \pm i\mathscr{E}_{y_2} = \frac{\delta_2{}^{\pm} K_1{}^{\pm} \{\exp(i\beta_1{}^{\pm}\xi) - \exp(i\beta_2{}^{\pm}\xi)\}}{(\beta_2{}^{\pm})^2 - (\beta_1{}^{\pm})^2} \tag{3.11}$$

in case (i) and

$$\mathscr{E}_{x_2} \pm i\mathscr{E}_{y_2} = \frac{\delta_2{}^{\pm} K_1{}^{\pm}}{2i\beta_1{}^{\pm}} \xi \exp(i\beta_1{}^{\pm}\xi) \tag{3.12}$$

in case (ii).

3.2. *Reflection*

Let an elliptically polarized electromagnetic wave having the electric vectors

$$\mathscr{E}_{x_1} \pm i\mathscr{E}_{y_1} = (A_{ix_1} \pm iA_{iy_1}) \exp\{i(\omega t - \xi)\}$$

be incident normally on the medium–free-space interface, viz. the plane $\xi = 0$ from the free-space side. The region $-\infty < \xi < 0$ corresponds to free space whereas $0 < \xi < \infty$ corresponds to the homogeneous medium.

The electric field in the free space is given by

$$\begin{aligned}(\mathscr{E}_x \pm i\mathscr{E}_y)_{\mathrm{F}} = {} & (A_{ix_1} \pm iA_{iy_1}) \exp\{i(\omega t - \xi)\} + (A_{rx_1} \pm iA_{ry_1}) \exp\{i(\omega t + \xi)\} \\ & + (A_{rx_2} \pm iA_{ry_2}) \exp\left[i\left\{(\omega + \Omega)t + \frac{\omega + \Omega}{\omega}\xi\right\}\right] \\ & + (A_{rx_3} \pm iA_{ry_3}) \exp\left[i\left\{(\omega - \Omega)t + \frac{\omega - \Omega}{\omega}\xi\right\}\right].\end{aligned} \tag{3.13}$$

(i) When $\beta_2{}^{\pm} \neq \beta_1{}^{\pm}$, the electric field in the medium is

$$\begin{aligned}(\mathscr{E}_x \pm i\mathscr{E}_y)_{\mathrm{M}} = {} & K_1{}^{\pm} \exp\{i(\omega t + \beta_1{}^{\pm}\xi)\} + K_2{}^{\pm} \exp[i\{(\omega + \Omega)t + \beta_2{}^{\pm}\xi\}] \\ & + \frac{\delta_2{}^{\pm} K_1{}^{\pm} \exp[i\{(\omega + \Omega)t + \beta_1{}^{\pm}\xi\}]}{(\beta_2{}^{\pm})^2 - (\beta_1{}^{\pm})^2} + K_3{}^{\pm} \exp[i\{(\omega - \Omega)t + \beta_3{}^{\pm}\xi\}] \\ & + \frac{\delta_3{}^{\bullet} K_1{}^{\pm} \exp[i\{(\omega - \Omega)t + \beta_1{}^{\pm}\xi\}]}{(\beta_3{}^{\pm})^2 - (\beta_1{}^{\pm})^2}.\end{aligned}$$

The boundary conditions to be applied are

$$\mathscr{E}_{\mathrm{F}}|_{\xi=0} = \mathscr{E}_{\mathrm{M}}|_{\xi=0} \tag{3.14a}$$

and

$$\left.\frac{\partial \mathscr{E}_{\mathrm{F}}}{\partial \xi}\right|_{\xi=0} = \left.\frac{\partial \mathscr{E}_{\mathrm{M}}}{\partial \xi}\right|_{\xi=0}. \tag{3.14b}$$

The amplitudes of the reflected components are found to be

$$A_{rx_1} \pm iA_{ry_1} = \frac{1 + \beta_1{}^{\pm}}{1 - \beta_1{}^{\pm}} (A_{ix_1} \pm iA_{iy_1}) \tag{3.15}$$

$$A_{rx_2} \pm iA_{ry_2} = -\frac{2\delta_2{}^{\pm}(A_{ix_1} \pm iA_{iy_1})}{(1 - \beta_1{}^{\pm})(\beta_2{}^{\pm} + \beta_1{}^{\pm})\{(\omega + \Omega)/\omega - \beta_2{}^{\pm}\}}. \tag{3.16}$$

(ii) When $\beta_2^{\pm} = \beta_1^{\pm}$, the electric vector in the medium is given by

$$(\mathscr{E}_x \pm i\mathscr{E}_y)_{\mathrm{M}} = K_1^{\pm} \exp\{i(\omega t+\beta_1^{\pm}\xi)\}+K_2^{\pm} \exp[i\{(\omega+\Omega)t+\beta_1^{\pm}\xi\}]$$
$$+\frac{\delta_2^{\pm}K_1^{\pm}}{2i\beta_1^{\pm}}\left(\xi-\frac{1}{2i\beta_1^{\pm}}\right) \exp[i\{(\omega+\Omega)t+\beta_1^{\pm}\xi\}]$$
$$+K_3^{\pm} \exp[i\{(\omega-\Omega)t+\beta_1^{\pm}\xi\}]$$
$$+\frac{\delta_3^{\pm}K_1^{\pm}}{2i\beta_1^{\pm}}\left(\xi-\frac{1}{2i\beta_1^{\pm}}\right) \exp[i\{(\omega-\Omega)t+\beta_1^{\pm}\xi\}].$$

If we apply the same boundary conditions as in (i), the amplitudes of the reflected components are

$$A_{\mathrm{r}x_1} \pm iA_{\mathrm{r}y_1} = \frac{1+\beta_1^{\pm}}{1-\beta_1^{\pm}}(A_{\mathrm{i}x_1} \pm iA_{\mathrm{i}y_1}) \tag{3.17}$$

$$(A_{\mathrm{r}x_2} \pm iA_{\mathrm{r}y_2}) = -\frac{\delta_2^{\pm}(A_{\mathrm{i}x_1} \pm iA_{\mathrm{i}y_1})}{\beta_1^{\pm}(1-\beta_1^{\pm})^2}. \tag{3.18}$$

The expressions corresponding to the frequency $\omega-\Omega$ are similar except for the difference that Ω is to be replaced by $-\Omega$.

4. Discussion

Equations (3.11) and (3.12) give us the magnitudes of the two modes of propagation of the sideband components generated in the propagating electromagnetic wave because of the presence of an alternating magnetic field. The corresponding expressions for the waves reflected from a plasma– or semiconductor–free-space interface are given by equations (3.16) and (3.18).

When both the extraordinary and the ordinary modes of the carrier as well as the two sidebands are present in a medium, it is difficult to define a unique modulation percentage; this is so because in this case the carrier and the two sidebands are actually travelling in the form of elliptically polarized waves and the magnitudes of their electric vectors at any point in space vary from instant to instant. This difficulty can be avoided by defining the modulation percentage of the two circularly polarized modes separately. Thus in one case we assume the extraordinary carrier mode (and hence the extraordinary sideband modes) to be absent and define a percentage modulation for the ordinary wave only and vice versa for the other case. In other words we define

$$\frac{\mu_+}{2} = \frac{\mathscr{E}_{x_2}+i\mathscr{E}_{y_2}}{\mathscr{E}_{x_1}+i\mathscr{E}_{y_1}} \qquad \text{for} \qquad \mathscr{E}_{x_2}-i\mathscr{E}_{y_2} = \mathscr{E}_{x_1}-i\mathscr{E}_{y_1} = 0$$

and

$$\frac{\mu_-}{2} = \frac{\mathscr{E}_{x_2}-i\mathscr{E}_{y_2}}{\mathscr{E}_{x_1}-i\mathscr{E}_{y_1}} \qquad \text{for} \qquad \mathscr{E}_{x_2}+i\mathscr{E}_{y_2} = \mathscr{E}_{x_1}+i\mathscr{E}_{y_1} = 0.$$

The modulation percentage of a wave which has propagated a distance ξ into the medium is then given by

$$\mu_{\pm} = \frac{\delta_2^{\pm}}{2i\beta_1^{\pm}}\xi \qquad \text{for case (ii).} \tag{4.1}$$

We have only chosen case (ii) ($\omega \gg \Omega$) for numerical calculations because of its simplicity and because it will nearly always be valid for microwave frequencies. It is noted from equations (4.1) that $\mu_{\pm}$ is directly proportional to ξ; this conclusion is physically understandable because after all it is the propagation of the electromagnetic wave through the anisotropic medium which is responsible for the modulation process. The above expression is not valid for values of ξ which correspond to large values of $\mu_{\pm}$ (say 0·15), because then the approximation that the magnitude of the generated sideband components are much smaller than the fundamental (which we have used in deriving expressions for the sideband components of the current density) will be violated.

Numerical calculations have been made for studying the variation of the modulation percentage with electron density and with the dependence of the electron collision frequency on electron velocity for some typical values of other parameters. The case $\omega \gg \nu$ is more appropriate to plasmas; table 1 illustrates the variation of μ_+ with ω_p^2/ω^2 and n for the conditions $\omega \ll \omega_B$, $\omega = \omega_B$ and $\omega > \omega_B$ when $z = 2{\cdot}5$ cm, $\omega_{B_1}/\omega = 1$, and when $\nu_0/\omega = 0{\cdot}01$, $\nu_0/\omega = 0{\cdot}1$ and $\nu_0/\omega = 0{\cdot}001$ respectively.

Table 1

ω_p^2/ω^2	Modulation percentage (μ_+): $\omega_B \ll \omega$, $\nu_0/\omega = 0{\cdot}01$, $n = 0$	$\omega_B = \omega$, $\nu_0/\omega = 0{\cdot}1$, $n = 0$	$\omega_B = \omega$, $\nu_0/\omega = 0{\cdot}1$, $n = 1$	$\omega_B \gg \omega$, $\nu_0/\omega = 0{\cdot}001$, $n = 0$
0·2	0·71	7·06	5·24	0·62
0·4	1·69	10·4	7·91	1·23
0·6	3·48	12·8	9·82	1·83
0·8	7·27	14·9	11·4	2·42

$\omega \gg \nu$, $\omega_{B_1}/\omega = 1$, $Z = 2{\cdot}5$ cm.

It is noted that the modulation percentage increases with the electron density and decreases with the power to which the electron velocity is raised in the expression for ν. Further, the modulation percentage passes through a maximum at gyroresonance and decreases on both sides—the decrease on the high-field side is more as is seen by the fact that very low values of ν_0/ω are needed to obtain appreciable percentage of modulation. The variation with collision frequency was also studied but has not been presented; it was found that when the collision frequency increases by one order of magnitude the modulation percentage decreases by one order. Table 2 presents the variation for the other case, viz. $\nu \gg \omega$ (which is applicable to semiconductors and laboratory plasmas), for $n = 1$. The conclusions are analogous in this case.

Table 2

Modulation percentage

ω_p^2/ω^2	μ_+
0·2	3·42
0·4	6·83
0·6	10·3
0·8	13·7

$\omega \ll \nu$, $\omega/\nu_0 = 0{\cdot}1$, $\omega_{B_1}/\omega = 1$, $z = 2\lambda$.

The magnitude of the percentage modulation in the wave reflected from the medium–free-space interface was also studied; this has, however, not been presented because these magnitudes were negligible.

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